Constructing $\mathbb{Q}$-Fano 3-folds à la Prokhorov & Reid

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Introduction

A **$\mathbb{Q}$-Fano 3-fold** X is a normal projective 3-dimensional variety over $\mathbb{C}$ with $-K_X$ ample, at worst $\mathbb{Q}$-factorial terminal singularities and Picard rank $\rho_X = 1$. The **($\mathbb{Q}$-Fano) index** of X is:

$$q_X := \max\{q \in \mathbb{Z}_{\geq 1} : \exists A \in \mathrm{Cl}(X), -K_X = qA\}$$

and, given a Weil divisor A for which $-K_X = q_X A$, we consider X to be **polarised by** A, i.e. with an embedding into weighted projective space given by Proj of the graded ring

$$R(X, A) = \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(kA)).$$

Some of the basic numerical invariants of (X, A) are the **codimension** of this embedding, the **index** q_X and the **degree** A^3.

The **graded ring database** contains a list of 1964 possible Hilbert series for a $\mathbb{Q}$-Fano 3-fold (X, A) of index ≥ 2. However it is not known if a $\mathbb{Q}$-Fano 3-fold actually exists with each Hilbert series.

Main result [2]

For each case in Table 1 we can construct a **Sarkisov link**:

$$\begin{tikzcd} & X' \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ X & & Y \end{tikzcd}$$

where σ is a divisorial extraction from a certain kind of **irreducible singular curve** $\Gamma \subset X$ and π is the **Kawamata blowdown** of a divisor $E' \subset X'$ to a terminal cyclic quotient singularity.

Table 1: The $\mathbb{Q}$-Fano 3-folds Y we can construct.

$\deg\Gamma$	E	X	Y
7	$\mathbb{P}(1,2,1)$	$\mathbb{P}^3$	$Y \subset \mathbb{P}(1^4, 2^2, 3)$
5	$\mathbb{P}(1,2,1)$	$X_2 \subset \mathbb{P}^4$	$Y \subset \mathbb{P}(1^5, 2^2, 3)$
14	$\mathbb{P}(1,3,2)$	$\mathbb{P}(1^3, 2)$	$Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$
9	$\mathbb{P}(1,2,3)$	$X_4 \subset \mathbb{P}(1^2, 2^2, 3)$	$Y \subset \mathbb{P}(1^2, 2^2, 3^2, 4, 5)$

The first two cases were **constructed by Prokhorov & Reid** [3]. We generalise their construction to get the remaining cases. We can also get two more examples, however in these cases Γ is necessarily **reducible** and hence Y will have **large Picard rank** $\rho_Y > 1$.

A generalisation of Prokhorov & Reid's construction

(1) Embed $E = \mathbb{P}(1, r, ra-1)$ into a $\mathbb{Q}$-Fano 3-fold (X, A) such that $A|_E = \mathcal{O}_E(r)$. The point of considering such an embedding is that E has a $\frac{1}{r}(1,-1)$ **type** A_{r-1} **singularity** $P \in E$ which is supported at a **smooth point** $P \in X$.

Lemma. *Suppose that (X, A) admits an embedding $E \subset X$ such that $E \in |eA|$ and $A|_E = \mathcal{O}_E(r)$. Then X is either of the form $\mathbb{P}(1,1,a,ra-1)$ or $X_{ra} \subset \mathbb{P}(1,1,a,ra-1,e)$. By an explicit classification there are precisely* **10 cases** *with terminal singularities.*

(2) Let $\Gamma \subset E \subset X$ be an irreducible curve of degree d passing through $P \in X$ which is contained in the smooth locus of X.

> **Key claim:** *If Γ is has an* **'appropriately singular'** *point at $P \in X$, then there exists a terminal divisorial extraction*
>
> $$\sigma\colon (F \subset X') \to (\Gamma \subset X)$$
>
> *such that σ induces an* **isomorphism** *$E' \cong E$, where E' is the birational transform of E.*

If σ exists then it is given by the blowup of the **symbolic powers** of the ideal sheaf $\mathcal{I}_{\Gamma/X}$. We take $\sigma\colon X' \to X$ to be the left-hand side of our Sarkisov link.

(3) If we make the clever choice $d = qr - 1$ then, following the 2-ray game that starts with σ, we find a **nef divisor** B' which is **numerically trivial** along E'. We check that the corresponding morphism $\pi\colon X \to Y$ contracts E' to a $\frac{1}{ra+r-1}(1, r, ra-1)$ singularity.

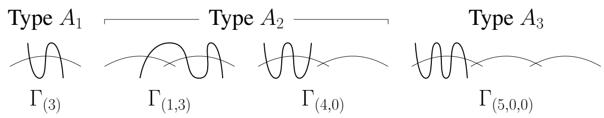
Conclusion. This construction is valid provided the divisorial extraction σ exists as in the **Key claim**. In this case we construct a Sarkisov link from (X, A) to a $\mathbb{Q}$-Fano 3-fold (Y, B) of index $q_Y = q - e$ and degree $B^3 = \frac{d}{ra+r-1}A^3$.

Divisorial extractions from singular curves

A type A_{r-1} Du Val singularity $P \in E$ has a resolution given by a **chain of (-2)-curves**. We call $P \in \Gamma \subset E$ a **curve singularity of type** $\Gamma_{(a_1,\dots,a_{r-1})}$ if the strict transform of Γ on this resolution is smooth with a_i branches intersecting the ith exceptional divisor transversely.

We now explain what **'appropriately singular'** means:

Proposition. *For the 10 cases found in the Lemma, $P \in E$ is either* **an A_1, A_2 or A_3 Du Val singularity**. *If a divisorial extraction σ from $\Gamma \subset X$ exists as in the* **Key claim**, *then $P \in \Gamma$ has one of the following singularity types (up to a degeneration):*



This follows from the unprojection method for constructing divisorial extractions explicitly [1] and by excluding cases according to $\deg\Gamma$.

Now we can check that the only cases admitting one of these singularity types (for Γ **irreducible**) are the four cases of Table 1. In the third case of Table 1 **both** of the A_2 singularity types are possible. In all other cases the singularity type is **unique**.

Unprojection construction for Y

We can construct Y explicitly using **unprojection**. Prokhorov & Reid [3] did this for the first two cases in Table 1. The third case $Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$ is interesting as there are **two possible constructions**, given by the **Tom and Jerry**. We have one 36-dimensional family of Tom unprojections and one 34-dimensional family of Jerry unprojections which correspond to Sarkisov links $Y \dashrightarrow \mathbb{P}(1^3, 2)$ ending in a contraction to a curve with singularity $\Gamma_{(1,3)}$ and $\Gamma_{(4,0)}$ respectively.

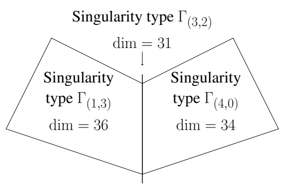


Figure 1: Two families of $Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$ $\mathbb{Q}$-Fano 3-folds.

They intersect in a 31-dimensional family, where the general member has a Sarkisov link $Y' \dashrightarrow \mathbb{P}(1^3, 2)$ ending in a contraction to a curve with singularity $\Gamma_{(3,2)}$ (a **common degeneration** of $\Gamma_{(1,3)}$ and $\Gamma_{(4,0)}$). However such Y' has a non-terminal singularity of index 1.

Further directions. Construct Sarkisov links with flips, flops and antiflips in the middle, or Sarkisov links which end with a Mori fibre space contractions, or a different type of divisorial contraction.

References

[1] T. Ducat, Divisorial extractions from singular curves in smooth 3-folds. *Int. J Math.*, **27**, Issue 01 (2016), 23 pp.

[2] T. Ducat, Constructing Q-Fano 3-folds à la Prokhorov & Reid, preprint, arxiv:1610.01773, 15 pp.

[3] Y. Prokhorov and M. Reid, On Q-Fano threefolds of Fano index 2, in *Minimal Models and Extremal Rays (Kyoto 2011)*, Adv. Stud. in Pure Math., **70**, 2016, 397–420